The Renormalisation Group (RG)

The goal of the RG is multiple :

1) Prove the scaling hypothesis
2) Highlight the concept of universality class
3) Give a procedure to compute the critical exponents

Let's start from the first and see how it interconnects with the others.

Step 1 : Coarse-graining

Coarse-graining, which we have already seen for the Ginzburg-Landau theory, provides a very natural way to change the lengthscale of a system.

Basically, we associate a new variable to groups of variables according to specific rules :

$$\{s_i\} \xrightarrow[g(\{s_i\},\{s_i'\})]{} \{s_i'\}$$

There are further constraint:

$$g(\{s_i\},\{s_j\}) = \begin{cases} 0 \\ 1 \end{cases}$$

$$\sum_{\{s_i'\}} g(\{s_i\},\{s_i'\}) = 1$$

Namely, for every combination of the original variables $\{s_i\}$ there is just one combination of the $\{s_i'\}$ variables that makes $g(\{s_i\},\{s_i'\}) = 1$, all the others corresponding to $g(\{s_i\},\{s_i'\}) = 0$.

With this tool in hand, we can now write, for an Ising model of N spins:

$$Z_N(T,h) = \sum_{\{s_i\}} e^{-\beta H(\{s_i\},T,h)}$$

$$= \sum_{\{s_i\}} e^{-\beta H(\{s_i\},T,h)} \cdot \boxed{\sum_{\{s_i'\}} g(\{s_i\},\{s_i'\})}_{=1} =$$

$$= \sum_{\{s_i'\}} \sum_{\{s_i\}} e^{-\beta H(\{s_i\},T,h)} \, g(\{s_i\},\{s_i'\})$$

We can now perform the sum over $\{s_i\}$:

$$Z_N(T,h) = \sum_{\{s_i'\}} e^{-\beta H'(\{s_i'\},T,h)}$$

where

$$H'(\{s_i'\},T,h) = -k_B T \ln\left[\sum_{\{s_i\}} e^{-\beta H(\{s_i\},T,h)} g(\{s_i\},\{s_i'\})\right]$$

from this formula we see that H' is actually a sort of free-energy.

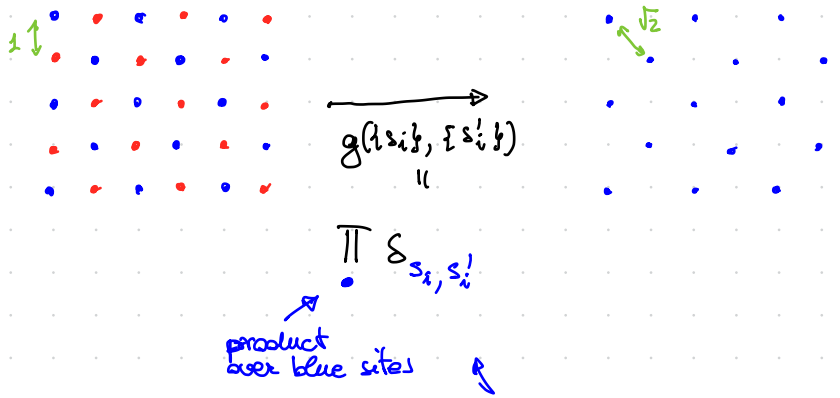
We have thus that

$$Z_N(T,h) = Z_{N'}(T,h)$$

↳ this is the partition function of a system defined by the new Hamiltonian $H'(\{s_i'\},T,h)$ and with N' degrees of freedom (the cardinality of $\{s_i'\}$)

In order to investigate better what this means we have better work through an example.

Ising in 2 dimensions :



We coarse-grain by using this definition of coarse-graining function. The sum over $\{s_i\}$ just substitutes the spins $\{s_i\}$ on the blue sites with the spins $\{s'_i\}$, while the spins on the red sites disappear.

We are left with a new lattice, only made by the blue sites, where the nearest neighbor distance is increased by a multiplicative factor $\sqrt{2}$.

Let's do the calculation explicitely.

The Ising Hamiltonian is:

$$H(\{s_i\}) = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$$

and the partition function is

$$Z_N(T,h) = \sum_{\{s_i\}} \sum_{\{s_i'\}} e^{\beta J \sum_{\langle i,j \rangle} s_i s_j + \beta h \sum_i s_i} \prod_i \delta_{s_i' s_i}$$

$$= \sum_{\{s_i'\}} e^{\beta h \sum_i s_i'} \sum_{\{s_i\}} e^{\beta J \sum_i \sum_{\text{n.n. of } i} s_i s_j' + \beta h \sum_i s_i} =$$

(nearest neighbors of i)

$$= \sum_{\{s_i'\}} e^{\beta h \sum_i s_i'} \sum_{\{s_i\}} e^{\sum_i s_i [\beta J \sum_{\text{n.n. of } i} s_j' + \beta h]} =$$

$$= \sum_{\{s_i'\}} e^{\beta h \sum_i s_i'} \prod_i \sum_{s_i = \pm 1} e^{s_i [\beta J \sum_{\text{n.n. of } i} s_j' + \beta h]} =$$

$$= \sum_{\{s_i'\}} e^{\beta h \sum_i s_i'} \prod_i \left[2 \cosh\left(\beta J \sum_{\text{n.n. of } i} s_j' + \beta h \right) \right] =$$

$$= 2^{N/2} \sum_{\{s_i'\}} e^{\beta h \sum_i s_i' + \sum_i \ln \cosh(\beta J \sum_{\text{n.n. of } i} s_j' + \beta h)} =$$

The system thus behaves as if it had a different Hamiltonian:

$$H' = -h \sum_i s_i' - k_B T \sum_i \ln \cosh\left(\beta J \sum_{n.n. \text{ of } i} s_j' + \beta h\right)$$

the spin s_i doesn't appear anymore. this i just marks the location of the n.n. of i

Is there a "better" way to write it?

$\ln\cosh(x)$ is a non-linear function of x that can be expanded in powers of x (let's set aside the issue of convergence).

This in turn requires all possible powers of $\sum_{n.n. \text{ of } i} s_j'$

but

$$(s_j')^{2n} = 1 \qquad (s_j')^{2n+1} = s_j'$$

as a consequence the power expansion of $\ln\cosh(\cdot)$ will contain only the terms

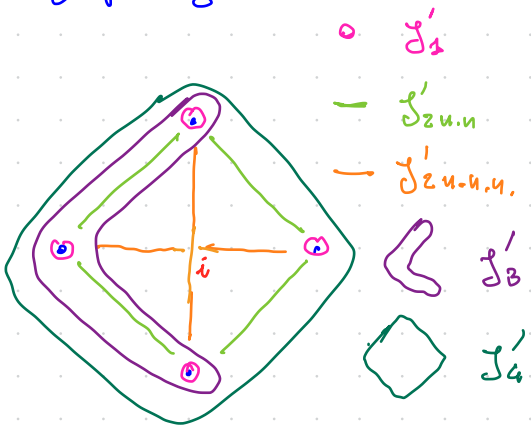
const	$\{s_j'\}$	$\{s_j' s_k'\}$	$\{s_j' s_k' s_\ell'\}$	$\{s_j' s_k' s_\ell' s_m'\}$
↑	↑	↑	↑	↑
1	4	6	4	1

and we can thus rewrite H' as

$$H' = \sum_i \left[J_0'(\beta,h) + J_1'(\beta,h) \sum_{n.n.i} s_j' + J_2'(\beta,h) \sum_{n.n.i} s_j' s_k' + J_2'(\beta,h) \sum_{n.n.n.i} s_j' s_k' + \right.$$
$$\left. + J_3'(\beta,h) \sum_{n.n.i} s_j' s_k' s_\ell' + J_4'(\beta,h) s_j' s_k' s_\ell' s_m' \right]$$

next nearest neighbors

Graphically:



What is interesting is that H and H' are different, but they have exactly the same partition function, hence they have exactly the same behavior.

Let us call now

$J_0 = 0 \quad J_1 = h \qquad J_{2\,n.n} = J \qquad J_{2,\,n.n.n} = 0$

$J_3 = 0$

$J_4 = 0$

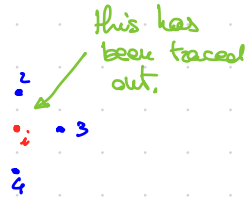
then the two Hamiltonians would have the same structure

$H(\{s_i\}, \{J_{ij}\}) \qquad H'(\{s_i\}, \{J_{ij}\})$

$$= 2^{N/2} \sum_{\{s_i'\}} e^{-\beta\left[-h\sum_i s_i' - k_B T \sum_i \ln\cosh\left(\beta J \sum_{\text{n.n. of } i} s_j' + \beta h\right)\right]}$$

this is a pretty non-linear function of $\{s_j'\}$

Let's count:



the group of four $\{s_i'\}$ around i → can take 16 values (2^4)

We would like to write it as a sum of polynomials of $\{s_i'\}$

4 linear terms: s_1', s_2', s_3', s_4'

$\binom{4}{2} = 6$ quadratic terms: $s_1's_2'$, $s_1's_3'$, $s_1's_4'$, $s_2's_3'$, $s_2's_4'$, $s_3's_4'$

$\binom{4}{3} = 4$ cubic terms: $s_1's_2's_3'$, $s_1's_2's_4'$, $s_1's_3's_4'$, $s_2's_3's_4'$

1 quartic term: $s_1's_2's_3's_4'$

1 constant (order 0 powers)

16

before writing down the equations, let's compute the number of parameters that we need

Let' simplify thing a bit further and call

$$\vec{K} = \beta \vec{J} \qquad \vec{K}' = \beta \vec{J}'$$

Then

$$\tilde{H} = \beta H$$

Now it is the same generalised Hamiltonian

$$Z_N(\vec{K}) = \sum_{\{s_i\}} e^{-\tilde{H}(\{s_i\}, \vec{K})} = 2^{N/2} \sum_{\{s_i'\}} e^{-\tilde{H}(\{s_i'\}, \vec{K}')} =$$

$$= 2^{N/2} Z_{N/2}(\vec{K}')$$

Let' say that we iterate the coarse-graining procedure. At the next step we are going to generate even more complicated interaction patterns.

It is called PROLIFERATION OF THE INTERACTIONS

In keeping with our previous writing, we define a vector $\vec{K}$ that contains the interaction strengths for all possible interaction patterns. In the thermodynamic limit it is an ∞-dimensional vector.

Equipped with this definition we can now see the coarse graining procedure as an iterative algorithm

$$\vec{K}^{(n+1)} = \vec{f}(\vec{K}^{(n)})$$

What is important is that once the problem is formulated in this way, with $\vec{K}$ being the vector of all possible interactions, then $\vec{f}$ is the same $\forall n$.

Once we have set up this iteration scheme, we can look for the usual feature of an iterative scheme : fixed points

$$\vec{K}^* = \vec{f}(\vec{K}^*) \qquad (\text{after } n \to \infty)$$

Let's first interpret $\vec{K}^*$:

1) we can expect that, after $n \to \infty$ iterations, $\vec{K}^*$ is full, even if it started from $\vec{K}^{(0)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

And recall that the physics of the system at $\vec{K}^*$ is exactly the same as the initial one.

Here we have a first lesson : all system with Hamiltonians $H(\vec{K}^{(n)})$ have the same physics.

There is an infinite family of models that behave in exactly the same way. It is a first idea of universality.

Actually, the idea is even stronger.

Typically, an iterative map has just a finite number of fixed points. This means that no matter the initial $\vec{K}$, the fixed point(s) will all be the same, and the fixed point

↑ there can be several

tells you what the physics of the system is.
There is an infinite family of systems with exactly the same behavior at the fixed point

2) What is the fixed point?
At the fixed point the Hamiltonian and the system does not change upon coarse-graining, that is upon rescaling.
What are the points in parameter space where the system is invariant upon rescaling?

$T=0$ $T=\infty$ and T_c

(for the Ising model)

Points (1) and (2) together tell us that, at the critical point, many systems that are microscopically diverse ($\vec{K}^{(0)}$ are different), at the critical points they all behave the same. It is the concept of

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Let's proceed further.

How do we analyze the iteration?

As usual, let's linearize close to it:

$$\vec{k}^{(n+1)} = \vec{k}^{*} + \delta\vec{k}^{(n+1)}$$

$$\vec{k}^{(n)} = \vec{k}^{*} + \delta\vec{k}^{(n)}$$

then

$$\vec{k}^{(n+1)} = \vec{k}^{*} + \delta\vec{k}^{(n+1)} = \vec{f}(\vec{k}^{*} + \delta\vec{k}^{(n)}) =$$

$$= \vec{f}(\vec{k}^{*}) + \underline{\underline{J}}_f^{*} \cdot \delta\vec{k}^{(n)}$$

$\vec{k}^{*}$

$$\Rightarrow \quad \delta\vec{k}^{(n+1)} = \underline{\underline{J}}_f^{*} \cdot \delta\vec{k}^{(n)}$$

where $\underline{\underline{J}}_f^{*}$ is the Jacobian matrix of $\vec{f}$ computed at the fixed point

This expression is analyzed by spectral expansion:

$$\underline{\underline{J}}_f^{*}\ \vec{v}_i = \lambda_i \vec{v}_i \qquad \vec{v}_i = \text{right eigenvector of eigenvalue } \lambda_i$$

$$\vec{u}_i\ \underline{\underline{J}}_f^{*} = \lambda_i \vec{u}_i \qquad \vec{u}_i = \text{left eigenvector of eigenvalue } \lambda_i$$

Then we write

$$\delta \vec{k}^{(n+1)} = \sum_i \xi_i^{(n+1)} \vec{v}_i \qquad \xi_i^{(n)} = \vec{u}_i \cdot \delta \vec{k}^{(n)}$$

and the relation

$$\delta \vec{k}^{(n+1)} = \underline{\underline{J}}^* \cdot \delta \vec{k}^{(n)}$$

becomes

$$\xi_i^{(n+1)} = \lambda_i \xi_i^{(n)}$$

There are two scenarios

a) $|\lambda_i| < 1$ b) $|\lambda_i| > 1$

(we neglect the very particular case $|\lambda_i| = 1$)

a)

$$|\xi_i^{(n+1)}| = |\lambda_i||\xi_i^{(n)}| = |\lambda_i|^2 |\xi_i^{(n-1)}| = |\lambda_i|^{n+1} |\xi_i^{(0)}|$$

$$\lim_{n \to \infty} |\xi_i^{(n)}| = 0$$

These are called irrelevant eigenvalues (directions)

b)

$$|\xi_i^{(n+1)}| = |\lambda_i||\xi_i^{(n)}| > |\xi_i^{(n)}|$$

relevant eigenvalues

Effectively we can thus write:

$$\tilde{H}(\vec{K}^{(n)}) \simeq \tilde{H}(\vec{K}^{*}; \{\xi_i^{(n)}\}_{\text{relevant}})$$

because the irrelevant one are infinitesimally small.
According to the iteration

$$\tilde{H}(\vec{K}^{(n+1)}) = \tilde{H}(\vec{K}^{*}, \{\xi_i^{(n+1)}\}_{\text{rel}}) =$$

$$= \tilde{H}(\vec{K}^{*}, \{\xi_i^{(n)} \lambda_i\}_{\text{rel}})$$

What can we say about λ_i ?

Let's connect the coarse-graining to a scale factor b.

$$\tilde{f}_b \longrightarrow \underline{\underline{J}}^{*}_{\vec{p},b} \longrightarrow \lambda_{i,b}$$

If we apply the coarse graining twice:

$$\xi_i^{(n+1)} = \lambda_{i,b}\, \xi_i^{(n)} = (\lambda_{i,b})^2\, \xi_i^{(n-1)}$$

but we can also think about a single coarse-graining operation that does exactly the same thing of twice the other. The scaling factor is then b^2:

$$\xi_i^{(n+1)} = \lambda_{i,b^2}\, \xi_i^{(n-1)}$$

But the two approaches must give the same result!

Then

$$(\lambda_{i,b})^2 = \lambda_{i,b^2}$$

We must express $\lambda_{i,b}$ as a function of the scaling factor:

$$\lambda_{i,b} = b^{y_i}$$

$$\hookrightarrow \quad (\lambda_{i,b})^2 = (b^{y_i})^2$$

$$\lambda_{i,b^2} = (b^2)^{y_i} \quad = !$$

We thus have

$$\xi_i^{(u+1)} = b^{y_i} \xi_i^{(u)}$$

We can now put everything together

$$\mathcal{Z}_N(\vec{K}^*, \{\xi_i^{(u)}\}_{rel}) = 2^{N'} \mathcal{Z}_{N'}(\vec{K}^*, \{\xi_i^{(u+1)}\}_{rel}) =$$

$$= 2^{N'} \mathcal{Z}_{N'}(\vec{K}^*, \{b^{y_i} \xi_i^{(u)}\}_{rel})$$

$N' = N/2$ in the case we have looked at, could be different.

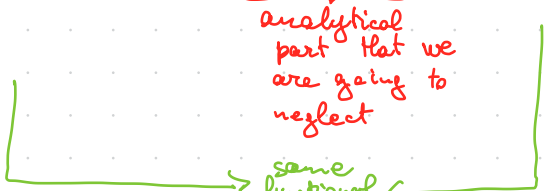
Important: $\mathcal{Z} = \sum e^{-\beta H}$

Here, from a functional point of view, the Hamiltonian is always the same, the parameters change, which means that the functional form of $\mathcal{Z}$ is always the same across the iterations

Take the logarithms:

$$-k_B T \ln \mathcal{Z}_N = F_N$$

$$\Rightarrow \quad F_N(\vec{K}^*, \{\xi_i^{(u)}\}) = \underbrace{-k_B T N' \ln 2}_{\text{analytical part that we are going to neglect}} + F_{N'}(\vec{K}^*, \{b^{y_i} \xi_i^{(u)}\})$$



The next step is to write

$$F_N = N f \qquad F_{N'} = N' f$$

Thus we have

$$N f(\vec{K}^*, \{\xi_i\}_{rel}) = N' f(\vec{K}^*, \{\xi_i b^{y_i}\}_{rel}) + \text{analytical}$$

then

$$f(\vec{K}^*, \{\xi_i\}_{rel}) = \frac{N'}{N} f(\vec{K}^*, \{\xi_i b^{y_i}\}_{rel}) + \dots$$

Upon rescaling of a factor b, we have $N'/N = b^{-d}$

(essentially: every volume containing 1 new degree of freedom contained, before coarse-graining, b^d degrees of freedom)

At last we have

$$f(\{\xi_i\}) = b^{-d} f(\{\xi_i b^{y_i}\})$$

We have almost proven the scaling hypothesis!

We still have to identify the relevant directions.

1) the only parameters that we can physically change are T and h.
2) the fixed point corresponds to the critical point
3) $\{\xi_i\}$ are small deviations with respect to the critical point

Then there are just two relevant directions, one associated with $|t|$ and the other with h.

Say

$$\xi_1 \propto |t|$$

$$\xi_2 \propto h$$

and at last we have

$$\boxed{f(|t|, h) = b^{-d} f(|t| b^{y_t}, h\, b^{y_h})}$$

On top of having proven the scaling hypothesis, we have also computed the critical exponents!

$$\lambda_{i,b} = b^{y_i} \Rightarrow \boxed{y_i = \frac{\ln \lambda_{i,b}}{\ln b} = \log_b \lambda_{i,b}}$$

$\lambda_{i,b}$ ← known from the Jacobian of $\vec{f}$

Truth: of course all of this is ideal, because we are not able to exactly know f, and $\underline{\underline{J}}_f$. But as an ideal proof, it is correct.

What have we learned?

The renormalisation group corresponds to an iterative procedure that converges to a fixed point.

If there are eigenvalues of the Jacobian matrix that are >1, the fixed point is unstable and the iterative coarse graining flows to the stable fixed points

fixed points:

- stable → fully ordered, fully disordered
- unstable → critical point

→ invariant under coarse-graining / rescaling: scale invariance